The m-spread model using branching processes

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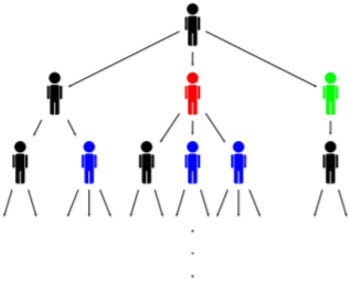
Motivation

When considering the evolution of a population, a branching process is often a good tool to describe it.

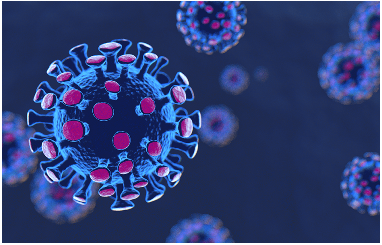
In a classical Galton-Watson branching process, the production of each individual is assumed to be independent of the past of the population.



Motivation

In reality, for example, when modeling the spread of a certain virus, the infection is more complicated.

The infectious period of the carriers usually last a while and before the recovery the individuals are still capable to pass on the viruses.



Motivation

Goals:

1. Construct a probability model to describe the evolution when the reproduction mechanism shows some dependence of the past of the population.

2. Study what happens to this kind of population in a long run.

Outline

Outline

Assumptions

Assume that all the individuals of the population are categorized into k different types.

Assume that the infectivity of each individual after infected will last for some time, say m units of time, and, during the infectious period, the individual has ability to pass the viruses to others.

Therefore, the type composition of the next generation depends on the type structure of the past m generations in the spread history.

The m-spread model

Let $\mathcal{A} = \{a_1, a_2, \cdots, a_k\}$ be the type set.

For $d \in \mathbb{N}$, we use the following notations:

1. T_d is the conventional d-tree.
2. $|g|$ stands for the length of $g \in T_d$, i.e., the number of the edges from the root ϵ of T_d to g.
3. Define $\Delta_n = \{g \in T_d : 0 \leq |g| \leq n\}$.
4. Define $\Delta_m^n = \Delta_n \backslash \Delta_m = \{g \in T_d : m < |g| \leq n\}$.
5. Let $F \subseteq \Delta_m$ and assume at least one $g \in F$ such that $|g| = m$. Every function

$$\alpha : F \to \mathcal{A}$$

is called an *m-order pattern* (or *m-pattern* for brevity).

The m-spread model

For example, $k = 2$, $\mathcal{A} = \{a, b\}$, $d = 2$ and $m = 4$:

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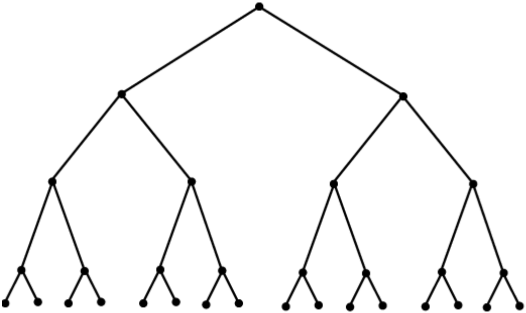

A 2-tree

The m-spread model

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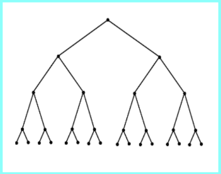

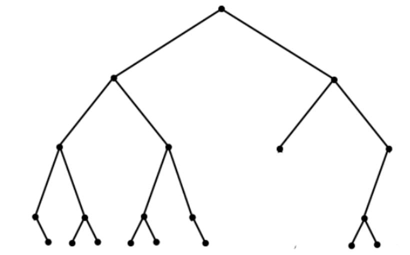

A subtree

The m-spread model

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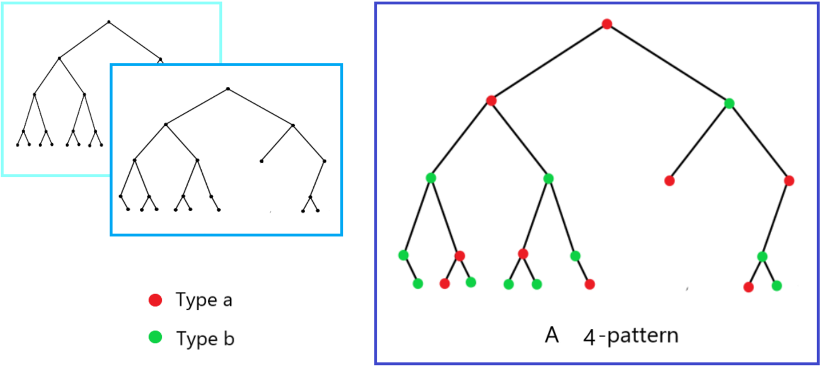


The m-spread model

Let $\mathcal{P}_n$ be the set of the collection of all the n-patterns. For each n-pattern $\alpha \in \mathcal{P}_n$, we define the following:

(i) The *parent* $\alpha^{(0)}$ of α, which is the subtree from the root of α up to the $(n-1)$th level in α.

(ii) For each $r = 1, 2, \cdots, n$, let $\Sigma_\alpha^{(r)}$ be the set of all the type-attached vertices in the rth level in α.

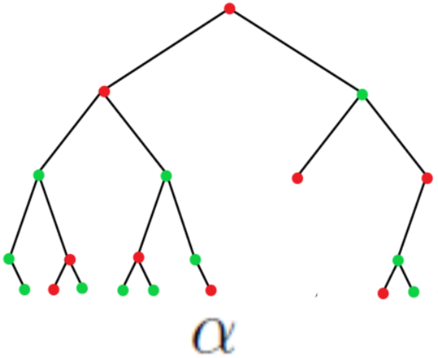
(iii) Let $\sigma_\alpha^{(r)} = |\Sigma_\alpha^{(r)}|$ be the number of vertices in $\Sigma_\alpha^{(r)}$. So, we can write

$$\Sigma_\alpha^{(r)} = \{g_1^{\alpha,r}, g_2^{\alpha,r}, \cdots, g_{\sigma_\alpha^{(r)}}^{\alpha,r}\}.$$

(iv) For each $r = 1, 2, \cdots, n-1$, $j = 1, 2, \cdots, \sigma_\alpha^{(r)}$, let $\alpha_j^{(r)}$ be the subtree rooted at $g_j^{\alpha,r}$ in the rth level of α and up to the nth level in α. We call $\alpha_j^{(1)}$ the jth "*son*" of α and $\alpha_j^{(r)}$ the jth "*r-descendant*" of α.

The m-spread model

For example, $d = 2$, $n = 4$ and $\mathcal{A} = \{a, b\}$.



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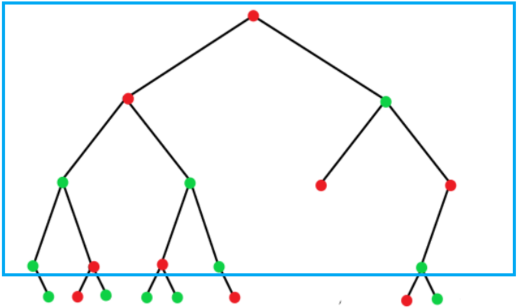

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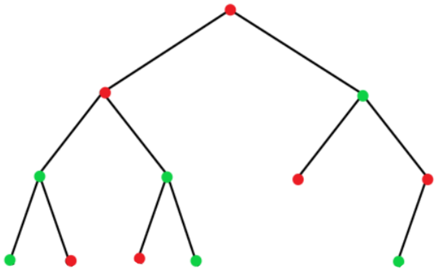
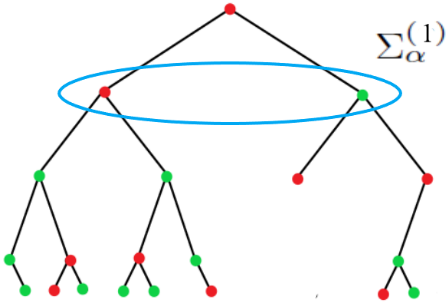


$\alpha^{(0)}$

The m-spread model

For example, $d = 2$, $n = 4$, $r = 1$ and $\mathcal{A} = \{a, b\}$.



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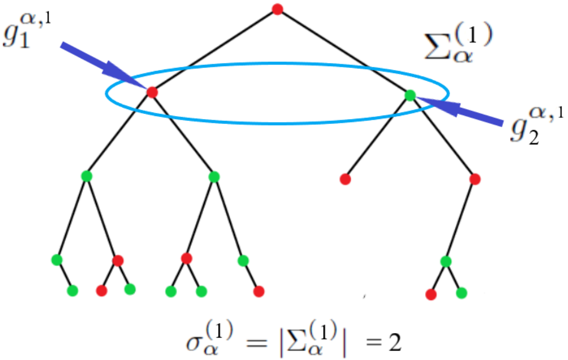

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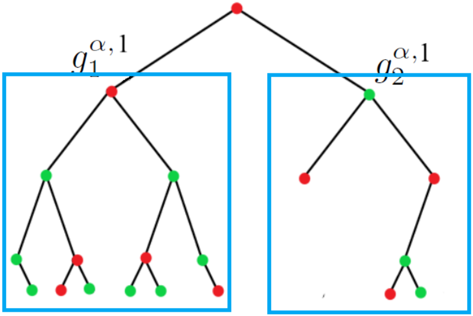

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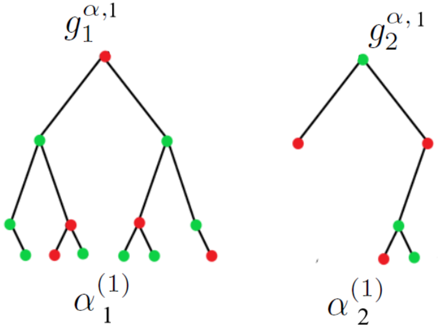


The m-spread model

Let $\mathcal{Q}_m$ be a subset of $\mathcal{P}_m$ such that, for every $\beta \in \mathcal{Q}_m$, there exist $q \in \mathcal{Q}_m$ such that

$$\beta_j^{(1)} = q^{(0)} \qquad \text{for all } j = 1, 2, \cdots, \sigma_\beta^{(1)}.$$

Since $\mathcal{P}_m$ is finite, so is $\mathcal{Q}_m$.

Let $\mathcal{Q}_m^{(0)} = \{\beta^{(0)} : \beta \in \mathcal{Q}_m\}$ be the collection of the parents of all m-patterns in $\mathcal{Q}_m$. We call $\mathcal{Q}_m^{(0)}$ the parent set of $\mathcal{Q}_m$.

Note that $\mathcal{Q}_m^{(0)} \subseteq \mathcal{P}_{m-1}$.

The m-spread model

Let $f : \mathcal{Q}_m^{(0)} \times \mathcal{Q}_m \to [0,1]$ be a function such that

(i) for any $\alpha \in \mathcal{Q}_m^{(0)}$ and any $\beta \in \mathcal{Q}_m$, if $\beta^{(0)} \neq \alpha$, then $f(\alpha, \beta) = 0$;

(ii) for each $\alpha \in \mathcal{Q}_m^{(0)}$, $\sum_{\beta \in \mathcal{Q}_m} f(\alpha, \beta) = 1$.

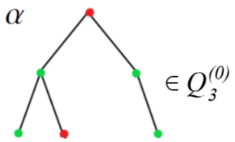
So, for any fixed $\alpha \in \mathcal{Q}_m^{(0)}$, $f(\alpha, \cdot) \equiv \{f(\alpha, \beta) : \beta \in \mathcal{Q}_m\}$ forms a probability distribution.

Each $f(\alpha, \beta)$ can be considered as the probability for the $(m-1)$-pattern α to grow into the m-pattern β.

The m-spread model

For example, $d = 2$, $\mathcal{A} = \{a, b\}$, $m = 3$, then

$$f : \mathcal{Q}_3^{(0)} \times \mathcal{Q}_3 \to [0, 1]$$



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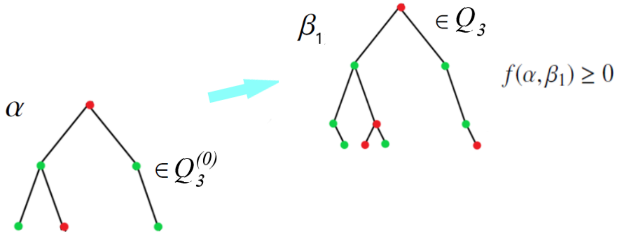

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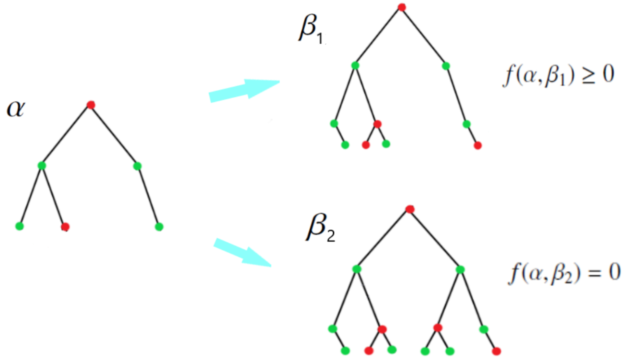


The m-spread model

Therefore, by Kolmogorov extension theorem, we can construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random elements

$$\{\zeta_\alpha : \alpha \in \mathcal{Q}_m^{(0)}\}$$

such that, for each α, ζ_α is a $\mathcal{Q}_m$-valued random element on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $(\zeta_\alpha)^{(0)}(\omega) = \alpha$ for each $\omega \in \Omega$ and the probability of ζ_α taking value at $\beta \in \mathcal{Q}_m$ is $f(\alpha, \beta)$.

The collection $\mathcal{R} = \{\zeta_\alpha : \alpha \in \mathcal{Q}_m^{(0)}\}$ of random m-patterns is called the *random m-spread model* over the type set $\mathcal{A}$ with pattern distribution f on $\mathcal{Q}_m^{(0)} \times \mathcal{Q}_m$.

Note that, if $\zeta_\alpha \in \mathcal{R}$ and $f(\alpha, \beta) > 0$ for $\beta \in \mathcal{Q}_m$, then

$$\zeta_{\beta_j^{(1)}} \in \mathcal{R} \qquad \text{for all } j = 1, 2, \cdots, \sigma_\beta^{(1)}.$$

The m-spread model

Assume that the first m generations of the population have existed initially and the type structure is α.

Let $\tau_\alpha^{m-1} = \alpha$, for some $\alpha \in \mathcal{Q}_m^{(0)}$ represent the initial type structure for the population from time (i.e.,generation) 0 up to time (i.e.,generation) $m-1$.

We will construct a sequence $\{\tau_\alpha^{m+n}\}_{n\geq 0}$ of random elements on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and each realization of $\{\tau_\alpha^{m+n}\}_{n\geq 0}$ can be considered as a growing tree and can be used to describe the evolution of the type structure of the population over time.

The m-spread model

Since $\alpha \in \mathcal{Q}_m^{(0)}$, it is the parent of some m-pattern in $\mathcal{Q}_m$.

So, at time 0, we can replace α with an m-pattern $\alpha_i^m \in \mathcal{Q}_m \equiv \{\alpha_i^m : 1 \leq i \leq l_m\}$ to obtain the value of the random element τ_α^m on $(\Omega, \mathcal{F}, \mathbb{P})$ with probability

$$\mathbb{P}(\tau_\alpha^m = \alpha_i^m) = f(\alpha, \alpha_i^m)$$

for all $i = 1, 2, \cdots, l_m$.

Note that τ_α^m has the same distribution as the random m-pattern ζ_α.

The m-spread model

We assume that the replacements of each $(m-1)$-pattern with an m-pattern is independent of the simultaneous replacement of other $(m-1)$-patterns.

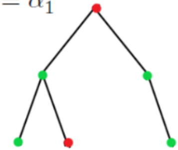
Since, if $\tau_\alpha^m = \alpha_{i_0}^m$, then $(\alpha_{i_0}^m)_j^{(1)} \in \mathcal{Q}_m^{(0)}$ for $j = 1, 2, \cdots, \sigma_{\alpha_{i_0}^m}^{(1)}$.
So, at time 1, we replace $(\alpha_{i_0}^m)_j^{(1)}$ with some $\alpha_{i_j}^m \in \mathcal{Q}_m$ to obtain a $(m+1)$-pattern $\alpha_i^{m+1} \in \mathcal{P}_{m+1}$ as the value of τ_α^{m+1} with probability

$$
\begin{aligned}
& \mathbb{P}(\tau_\alpha^{m+1} = \alpha_i^{m+1} | \tau_\alpha^m = \alpha_{i_0}^m) \\
= \; & \prod_{j=1}^{\sigma_{\alpha_{i_0}^m}^{(1)}} \mathbb{P}\big((\tau_\alpha^{m+1})_j^{(1)} = \alpha_{i_j}^m | \tau_\alpha^m = \alpha_{i_0}^m\big) = \prod_{j=1}^{\sigma_{\alpha_{i_0}^m}^{(1)}} f\big((\alpha_{i_0}^m)_j^{(1)}, \alpha_{i_j}^m\big).
\end{aligned}
$$

The m-spread model

For example, $d = 2$, $\mathcal{A} = \{a, b\}$, $m = 3$. Assume that initially we have $\alpha = \alpha_1^2 \in \mathcal{Q}_3^{(0)}$.

$\tau_\alpha^2 = \alpha_1^2$



The m-spread model

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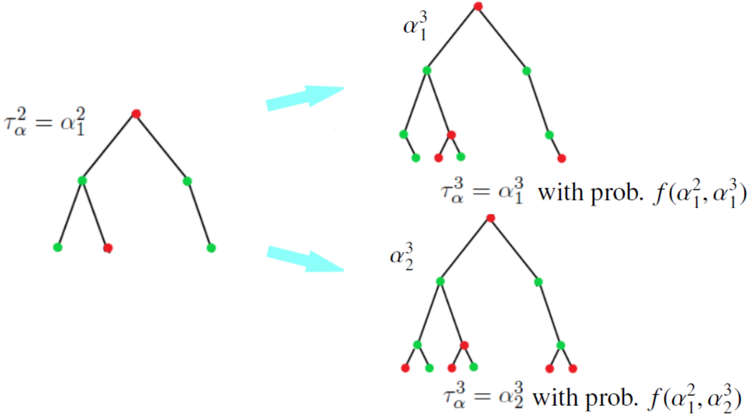

The m-spread model

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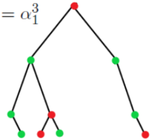
$\tau_\alpha^3 = \alpha_1^3$



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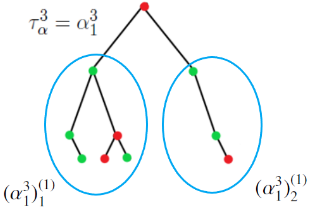

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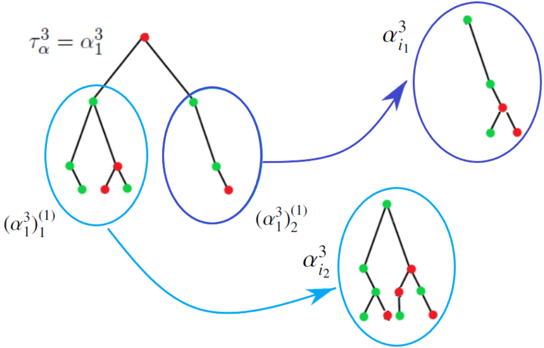

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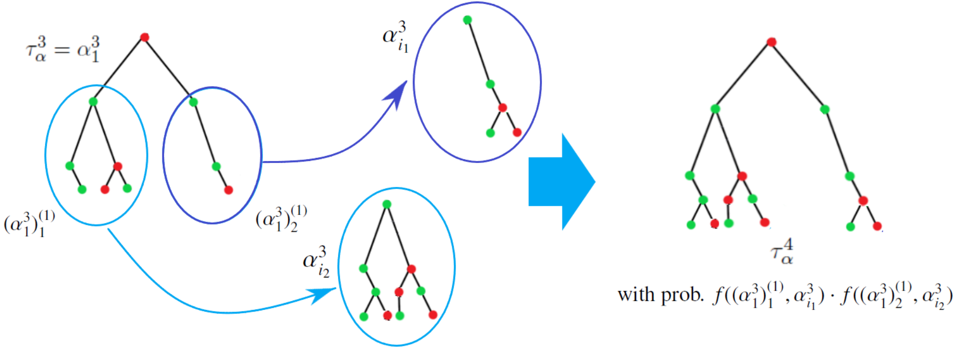

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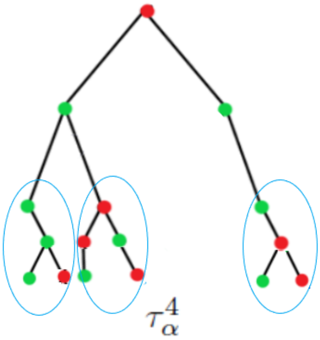


The m-spread model

It should be noted that the type structure in the first $m+1$ generations of τ_α^m and τ_α^{m+1} are identical with probability 1.

Therefore, we can consider that τ_α^m is growing into τ_α^{m+1}. Here, τ_α^{m+1} is a random element taking values in $\mathcal{P}_{m+1}$.

Assume that τ_α^{m+n} is constructed and note that τ_α^{m+n} takes values in $\mathcal{P}_{m+n}$.

Given $\tau_\alpha^{m+n} = \alpha_{i_0}^{m+n} \in \mathcal{P}_{m+n}$, we then replace each $(m-1)$-pattern

$$(\alpha_{i_0}^{m+n})_j^{(n+1)} \in \mathcal{Q}_m^{(0)}$$

which is rooted at $g_j^{\alpha_{i_0}^{m+n}, n+1}$, for $j = 1, 2, \cdots, \sigma^{(n)}_{\alpha_{i_0}^{m+n}}$,

The m-spread model

with some $\alpha_{i_j}^m \in \mathcal{Q}_m$ to obtain a $(m+n+1)$-pattern α_i^{m+n+1} as the value of τ_α^{m+n+1} with probability

$$\begin{aligned} & \mathbb{P}(\tau_\alpha^{m+n+1} = \alpha_i^{m+n+1} | \tau_\alpha^{m+n} = \alpha_{i_0}^{m+n}) \\ = & \prod_{j=1}^{\sigma^{(n+1)}_{\alpha_{i_0}^{m+n}}} \mathbb{P}\Big((\tau_\alpha^{m+n+1})_j^{(n+1)} = \alpha_{i_j}^m | \tau_\alpha^{m+n} = \alpha_{i_0}^{m+n}\Big) \\ = & \prod_{j=1}^{\sigma^{(n+1)}_{\alpha_{i_0}^{m+n}}} f\Big((\alpha_{i_0}^{m+n})_j^{(n+1)}, \alpha_{i_j}^m\Big), \end{aligned}$$

The m-spread model

We continue this process and obtain a sequence $\{\tau_\alpha^{m+n}\}_{n\geq 0}$ of almost surely growing random trees.

Therefore, the limit $\tau_\alpha \equiv \lim_{n\to\infty} \tau_\alpha^{m+n}$ exists with probability 1.

The random element τ_α takes values on $\mathcal{A}^{\bar{d}}$, where $\bar{d} = \max_{\beta\in\mathcal{Q}_m}\big(\max_{0\leq i\leq m}\sigma_\beta^{(i)}\big)$.

For almost every $\omega \in \Omega$, $\tau_\alpha^{m+n}(\omega)$ is the subtree from the root of $\tau_\alpha(\omega)$ to the $(m+n)$th level of $\tau_\alpha(\omega)$, for all $n = 0, 1, 2, \cdots$.

We call τ_α the *infinite random spread pattern induced from the random m-pattern ζ_α with respect to the random m-spread model $\mathcal{R}$ and pattern distribution f.*

The m-spread model

Question: If we go to the nth level of the infinite tree τ_α, what is the proposition of a given type a_i as $n \to \infty$?

More generally, if $\mathcal{O}_a(\tau_\alpha|_{\Delta_r^s(\tau_\alpha)})$ is the occurrences of $a \in \mathcal{A}$ in τ_α from level $r+1$ to level s and $\{k_n\}_{n=1}^\infty$ is an increasing sequence such that $k_n \to \infty$ as $n \to \infty$, then what happens to the rate:

$$\begin{aligned} & s_\alpha(a; \{k_n\}_{n=1}^\infty) \\ := \quad & \lim_{n\to\infty} s_\alpha(a; [s_n, s_{n+1}]) := \lim_{n\to\infty} \frac{\mathcal{O}_a(\tau_\alpha|_{\Delta_r^s(\tau_\alpha)})}{|\Delta_r^s(\tau_\alpha)|} \end{aligned}$$

where $s_n = \sum_{i=1}^{n} k_i$.

Outline

Main results

Recall that τ_α is the infinite spread pattern induced from the random m-pattern ζ_α with respect to the random m-spread model $\mathcal{R}$ and the type set $\mathcal{A} = \{a_i\}_{i=1}^{k}$.

Let $\mathbf{A} \equiv \mathcal{Q}_m^{(0)} = \{\alpha_i\}_{i=1}^{\mathbf{k}}$.

For each $a \in \mathcal{A}$, we define

$$\theta(a) = \{\beta \in \mathbf{A} : \epsilon(\beta) = a\}$$

where $\epsilon(\beta)$ is the root of the pattern β.

Main results

Theorem 1 (Ban, Hong and Wu, 2022+).

Let τ_α be the infinite spread pattern with the type set $\mathcal{A} = \{a_i\}_{i=1}^{k}$. Suppose that $\{k_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. Let $s_n = \sum_{r=1}^{n} k_n$. Then, for any $a_j \in \mathcal{A}$, we have that

$$s_\alpha(a_j; \{k_n\}_{n=1}^{\infty}) = \lim_{n\to\infty} \frac{\mathcal{O}_{a_j}\left(\tau_\alpha\big|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)}\right)}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)\right|} = \sum_{\substack{i=1,2,\cdots,\mathbf{k} \text{ s.t.} \\ \alpha_i \in \theta(a_j)}} \mathbf{v}_i$$

with probability 1, where $\vec{\mathbf{v}} = (\mathbf{v}_1, \cdots, \mathbf{v}_{\mathbf{k}})$ is the left eigenvector of the offspring mean matrix $\mathbf{M}$ of the induced branching process $\{\vec{\mathbf{Z}}_n\}_{n=0}^{\infty}$ associated with its maximal eigenvalue.

Main results

Theorem 2 (Ban, Hong and Wu, 2022+).

Under the hypotheses of Theorem 1, if $k_n = kn$ for all $n \in \mathbb{N}$, then, we have that

$$s_\alpha(a_j; k) \equiv \lim_{n\to\infty} \frac{\mathcal{O}_{a_j}\left(\tau_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau_\alpha)}\right)}{\left|\Delta_{kn}^{k(n+1)}(\tau_\alpha)\right|} = \sum_{\substack{i=1,2,\cdots,\mathbf{k} \text{ s.t.} \\ \alpha_i \in \theta(a_j)}} \mathbf{v}_i$$

with probability 1.

In particular, if $k = 1$, then $\mathcal{O}_a(\tau_\alpha|_{\Delta_{n-1}^{n}(\tau_\alpha)})$ is the occurrences of the type a in the nth generation of τ_α.

Outline

Induced Branching Process

We will construct a multitype branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in which each $(m-1)$-pattern is considered as a type.

Let $\mathbf{A} = \mathcal{Q}_m^{(0)} = \{\alpha_1, \alpha_2, \cdots, \alpha_{\mathbf{k}}\}$.

If, in the original population, $\tau_\alpha^{m-1} = \alpha = \alpha_{i_0}$ for some $\alpha_{i_0} \in \mathbf{A}$, then we let the induced population $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ start with an individual of type $\alpha_{i_0} \in \mathbf{A}$ at time 0, i.e., $\vec{\mathbf{Z}}_0 = \vec{e}_{i_0}$, where $\vec{e}_{i_0}$ is the standard unit vector with 1 as its i_0th component and 0 elsewhere.

We also write $\{\vec{\mathbf{Z}}_n^{(i_0)}\}_{n\geq 0}$ for $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ when $\vec{\mathbf{Z}}_0 = \vec{e}_{i_0}$.

Induced Branching Process

After one unit of time, when the type structure of the population in the m-spread model grows from τ_α^{m-1} to τ_α^m, at the same time, in the induced population, we replace the initial $(m-1)$-pattern α_{i_0} with $\sigma_{\tau_\alpha}^{(1)}$, the $(m-1)$-patterns rooted at level 1 in τ_α^m, to obtain the population

$$\left\{(\tau_\alpha^m)_1^{(1)}, (\tau_\alpha^m)_2^{(1)}, \cdots, (\tau_\alpha^m)_{\sigma_{\tau_\alpha}^{(1)}}^{(1)}\right\}$$

at time 1, which are called the *children* of the initial $(m-1)$-pattern in the language of branching process.

Note that each $(\tau_\alpha^m)_i^{(1)}$ is a random element taking values in the type set $\mathbf{A}$.

Induced Branching Process

Let $\mathbf{Z}_{1,i}^{(i_0)}$ be the number of the $(m-1)$-patterns of type α_i in the population at time 1, $i = 1, 2, \cdots, \mathbf{k}$, and call the vector $\vec{\mathbf{Z}}_1^{(i_0)} = (\mathbf{Z}_{1,1}^{(i_0)}, \mathbf{Z}_{1,2}^{(i_0)}, \cdots, \mathbf{Z}_{1,\mathbf{k}}^{(i_0)})$ the population vector at time 1.

Next, after another unit of time, the type structure of the population grows from τ_α^m to τ_α^{m+1}. So, again in the induced population, we replace $\sigma_{\tau_\alpha}^{(1)}$ $(m-1)$-patterns rooted at individuals in the 1st generation with $\sigma_{\tau_\alpha}^{(2)}$ $(m-1)$-patterns

$$\left\{(\tau_\alpha^{m+1})_1^{(2)}, (\tau_\alpha^{m+1})_2^{(2)}, \cdots, (\tau_\alpha^{m+1})_{\sigma_{\tau_\alpha}^{(2)}}^{(2)}\right\}$$

rooted at individuals in the 2nd generation in τ_α^{m+1} to obtain the population vector $\vec{\mathbf{Z}}_2^{(i_0)} = (\mathbf{Z}_{2,1}^{(i_0)}, \mathbf{Z}_{2,2}^{(i_0)}, \cdots, \mathbf{Z}_{2,\mathbf{k}}^{(i_0)})$ at time 2 and so on.

Induced Branching Process

Let $\vec{\mathbf{Z}}_n^{(i_0)} = (\mathbf{Z}_{n,1}^{(i_0)}, \mathbf{Z}_{n,2}^{(i_0)}, \cdots, \mathbf{Z}_{n,\mathbf{k}}^{(i_0)})$ be the population vector at time n, where $\mathbf{Z}_{n,i}^{(i_0)}$ is the number of the $(m-1)$-patterns of type α_i among the population

$$\left\{ (\tau_\alpha^{m+n-1})_1^{(n)}, (\tau_\alpha^{m+n-1})_2^{(n)}, \cdots, (\tau_\alpha^{m+n-1})_{\sigma_{\tau_\alpha}^{(n)}}^{(n)} \right\}$$

at time n.

Then such a process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ is called the *induced process* with the induced type set $\mathbf{A}$ from the random m-spread model $\mathcal{R}$.

Induced Branching Process

From the construction, for almost every realization of the infinite spread pattern τ_α, there is a corresponding realization of the process $\{\vec{\mathbf{Z}}_n\}$ and, we directly obtain the following properties:

(i) Each $(m-1)$-pattern rooted at a node in the nth level of τ_α represents an individual in the nth generation of $\{\vec{\mathbf{Z}}_n\}$.

(ii) If $\alpha = \alpha_{i_0} \in \mathbf{A}$, then, for all $n = 0, 1, 2, \cdots$, $|\vec{\mathbf{Z}}_n^{(i_0)}| = \sigma_{\tau_\alpha}^{(n)}$ with probability 1.

Induced Branching Process

(iii) The induced process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ from any random m-spread model $\mathcal{R}$ is a multi-type Galton-Watson branching process with the type set $\mathbf{A}$ and the offspring distribution $\{\mathbb{P}^{(i)}(\cdot)\}_{i=1}^{\mathbf{k}}$, where

$$\mathbb{P}^{(i_0)}(\vec{r}) = \sum_{\substack{\beta\in\mathcal{Q}_m \text{ s.t.} \\ \mathbf{T}(\beta)=\vec{r}}} f(\alpha_{i_0}, \beta)$$

and $\mathbf{T}(\beta) = (t_1, t_2, \cdots, t_{\mathbf{k}})$ is the vector with t_i as the number of the $(m-1)$-patterns of type α_i among $\left\{\beta_1^{(1)}, \beta_2^{(1)}, \cdots, \beta_{\sigma_\beta^{(1)}}^{(1)}\right\}$, $i = 1, 2, \cdots, \mathbf{k}$.

(iv) $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ is positive regular.

Induced Branching Process

We further assume that the induced branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ is non-singular.

By the well-known limit theorems for classical multitype branching processes, we have the growth rate regarding the induced branching process:

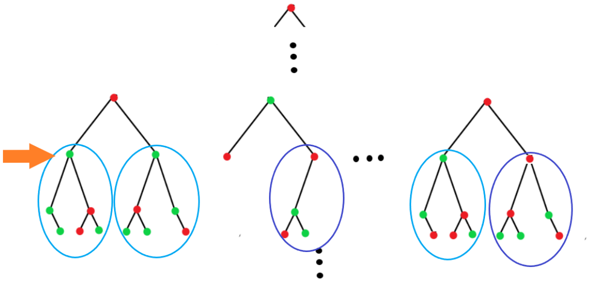
Induced Branching Process

Lemma 3.

Let $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ *be the induced branching process from the random* m*-spread model* $\mathcal{R}$*. Let* $\mathbf{M}$ *be the offspring mean matrix with the spectral radius* $\rho > 1$ *and the corresponding right and left eigenvectors* $\vec{\mathbf{u}} = (\mathbf{u}_1, \cdots, \mathbf{u}_\mathbf{k})$ *and* $\vec{\mathbf{v}} = (\mathbf{v}_1, \cdots, \mathbf{v}_\mathbf{k})$ *such that* $\vec{\mathbf{v}} \cdot \vec{\mathbf{1}} = 1$ *and* $\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = 1$*. Then, for every* $i = 1, 2, \cdots, \mathbf{k}$,

$$s_{\alpha_i}(\alpha_j) = \lim_{n\to\infty} s_{\alpha_i}^{(n)}(\alpha_j) \equiv \lim_{n\to\infty} \frac{\left| \prod(\mathbf{p}_{\alpha_i}^{(n)}) \right|_{\alpha_j}}{\sigma_{\alpha_i}^{(n)}} = \mathbf{v}_j \quad \textit{w.p.1.}$$

Induced Branching Process



$$s_\alpha(a_j;\{k_n\}_{n=1}^{\infty}) = \lim_{n\to\infty} \frac{\mathcal{O}_{a_j}\left(\tau_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)}\right)}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)\right|} = \sum_{\substack{i=1,2,\cdots,\mathbf{k} \text{ s.t.} \\ \alpha_i\in\theta(a_j)}} \mathbf{v}_i$$

References

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3. Jung-Chao Ban, Jyy-I Hong and Yu-Liang Wu, *The m-spread models*, preprint.
4. Image credit to https://www.scmp.com /news/china/society /article /3076323 /third-coronavirus-cases-may-be-silent-carriers-classified

Thank you